1. Exercises from Sections 2.9

PROBLEM 1. *(Folland 2.9.1) Find the extreme values of* $f(x,y) = 3x^2 - 2y^2 + 2y$ *on the set* $\{(x,y)\,|\,x^2+y^2 \le 1\}$

- Extreme values can occur either on the boundary, or at critical points on the interior
- On the interior of the disk, we can calculate the gradient:

$$\nabla f = 6x\partial_x + (2-4y)\partial_y = 0 \Longleftrightarrow x = 0, y = 1/2$$

- And the Hessian is given by:

$$H = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$$

- The critical point at $(0,1/2)$ is a saddle because the Hessian has one negative eigenvalue and one positive one. This point could not be an extreme value, therefore the extrema of $f(x,y)$ occur on the boundary.
- On the boundary $\{(x,y)\,|\,x^2+y^2 = 1\}$ we have $f(x,y) = 3(1-y^2) - 2y^2 + 2y = -5y^2 + 2y + 3$
- This function is maximized at $y = 1/5$, $x = \pm\sqrt{24/25}$
- To find the minima, just check the values at $y = \pm 1$

$$f(x(1),1) = -5+2+3 = 0$$

$$f(x(-1),-1) = -5-2+3 = -4$$

- Therefore $f(x,y)$ is minimized at $x = 0, y = -1$.

PROBLEM 2. *Show that* $f(x,y) = (x^2+2y^2)e^{-x^2-y^2}$ *has an absolute minimum and maximum on* $\mathbb{R}^2$*, then find them*

- The exponential decay dominates the polynomial growth of $f(x,y)$ as $r \to \infty$

$$\lim_{(x,y)\to\infty} |(x^2+2y^2)e^{-x^2-y^2}| = \lim_{(x,y)\to\infty} x^2e^{-x^2-y^2} + \lim_{(x,y)\to\infty} 2y^2e^{-x^2-y^2} \le \lim_{(x,y)\to\infty} x^2e^{-x^2} + \lim_{(x,y)\to\infty} 2y^2e^{-y^2} = 0$$

- So $\lim_{r\to\infty} f(x,y) = 0$.
- Also, it is clear that $f(x,y) \ge 0$, therefore by theorem 2.83 the function $f(x,y)$ has a maximum
- We find the maximum by locating the critical points

$$\begin{aligned} \nabla f &= \left[2xe^{-x^2-y^2} + e^{-x^2-y^2}(-2x)(x^2+2y^2)\right]\partial_x + \left[4y^{-x^2-y^2} + e^{-x^2-y^2}(-2y)(x^2+2y^2)\right]\partial_y \\ &= \left(2-2x^2-4y^2\right)xe^{-x^2-y^2}\partial_x + \left(4-2x^2-4y^2\right)ye^{-x^2-y^2}\partial_y \end{aligned}$$

- We'll also need the second partials, so let's calculate those as well:

$$f_{xx} = (2-6x^2-4y^2)e^{-x^2-y^2} + (2x-2x^3-4y^2x)(-2x)e^{-x^2-y^2}$$

$$f_{yy} = (4-2x^2-12y^2)e^{-x^2-y^2} + (4y-2x^2y-4y^3)(-2y)e^{-x^2-y^2}$$

$$f_{xy} = f_{yx} = -8xye^{-x^2-y^2} + (2x-2x^3y-4y^2x)(-2y)e^{-x^2-y^2}$$

- Look for (x,y) such that $\nabla f = 0$ (notice the exponential factor is stricly positive, so we can ignore it). Split into cases:
- Case 1: $(x,y) = (0,0)$

$$H^{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Since $\det H = \alpha\gamma - \beta^2 > 0$ and $\alpha = f_{xx} > 0$, $(0,0)$ is a minimum

- Case 2: $y = 0$, $2 - 2x^2 - 4y^2 = 0 \Longrightarrow x = \pm 1$
$$H^{(\pm 1,0)} = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$$
Since $\det H = \alpha\gamma - \beta^2 < 0$, $(\pm 1, 0)$ are both saddle points
- Case 3: $x = 0$, $4 - 2x^2 - 4y^2 = 0 \Longrightarrow y = \pm 1$
$$H^{(0,\pm 1)} = \begin{pmatrix} -2 & 0 \\ 0 & -8 \end{pmatrix}$$
Since $\det H = \alpha\gamma - \beta^2 > 0$ and $\alpha = f_{xx} < 0$, $(0, \pm 1)$ is a minimum
- Case 4: $2 - 2x^2 - 4y^2 = 0$ and $4 - 2x^2 - 4y^2 = 0$. This is impossible, since concentric ellipses do not intersect.

PROBLEM 3. *(Folland 2.9.19) Let A be a symmetric $n \times n$ matrix, and let $f(x) = x^T Ax$ for $x \in \mathbb{R}^n$. Show that the maximum and minimum of f on the unit sphere $|x|^2 = 1$ are the largest and smallest eigenvalues of A.*

(1) Proceed by method of Lagrange multipliers
(2) Want to find extrema of $f(x)$ subject to the constraint $|x|^2 = 1$
(3) Write $f(x) = \sum_{ij} A_{ij} x_i x_j$ and let $G(x) = 1 - \sum_i x_i^2$
(4) The condition to optimize f subject to G is given by:
$$\nabla f = \lambda \nabla G$$
(5)
$$\nabla f = \partial_k f = \sum_{ij} (A_{ij}\delta_{ik}x_j + A_{ij}\delta_{jk}x_i) = \sum_j A_{kj}x_j + \sum_i A_{ik}x_i = 2\sum_j A_{kj}x_j = 2Ax$$
$$\nabla G = \partial_k G = \sum_i 2x_i \delta_{ik} = 2x_k$$
(6) Then the condition for extremizing f is that $Ax = \lambda x$
(7) Since A is symmetric, it is diagonalizable, so pick an orthonormal basis $\{y_j\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A; so $Ay_j = \lambda_j y_j$.
(8) In this basis, we may write $x = \sum_j c_j y_j$, so that $f(x) = x^T Ax = \sum_i c_i^2 \lambda_i$
(9) Now it is clear that the extrema have values $f(y_j) = \lambda_j$, and thus the global max and min of f are the largest and smallest eigenvalues of A, respectively.